THE DIFFERENT MATHEMATICAL APPROXIMATION FORMATS FOR THE FIELD EMISSION SPECIAL MATHEMATICAL FUNCTION v(x)

Richard G. Forbes
Advanced Technology Institute & Department of Electrical and Electronic Engineering,
University of Surrey, Guildford, Surrey GU2 7XH, UK
Permanent e-mail alias: r.forbes@trinity.cantab.net

Given the virtual nature of this conference, it has seemed most useful to present the additional material (beyond that in the Abstract) that would have been on an A0 Poster as a supplement to this abstract, in the "COMPARISONS" section below.

ABSTRACT

Within the context of smooth-planar-metal-emitter (SPME) theoretical methodology, the best simple model for describing the field electron emission (FE) tunnelling barrier is the Schottky-Nordheim (SN) ("planar image rounded") barrier. Use of this barrier within the context of the simple (or "first-order") JWKB mathematical tunnelling formalism leads to so-called Murphy-Good-type FE equations. For experiment-facing theory, the author now prefers to use a so-called *Extended Murphy-Good (EMG) FE Equation* to describe the relationship between the emission current I_e and the emitter's characteristic local barrier field F_C and local work function ϕ (usually taken as the emitter-apex values). This EMG equation has the form

$$I_e^{EMG} = A_f^{SN}\, a\phi^{-1}F_C^{2}\exp[-v_F b\phi^{3/2}/F_C],$$

where a and b are the Fowler-Nordheim constants, A_f^{SN} is the formal emission area for the SN barrier and v_F is a particular value (appropriate to a barrier described by ϕ and F_C) of the principal field emission special mathematical function v(x). Here, v(x) is expressed as a function of the *Gauss variable* x, which is my name for the independent variable in the Gauss Hypergeometric Differential Equation. The function v(x) is a special solution of this equation, and it can be shown that v(x) is applied to FE theory by setting v_F=v(x=f_C), where the *characteristic scaled field* f_C is related to the barrier field F_C by $f_C \equiv (e^3/4\pi\varepsilon_0)\phi^{-2}F_C$.

We now know that v(x) is an unusual mathematical function in that its efficient representation as an exact series expansion requires TWO infinite power series, rather than one. Hence, efficient representation CANNOT be obtained by simple Taylor expansion.

In FE literature there exist many (around 20) different approximate mathematical formulae for v_F, mostly expressed as functions of f_C, or more commonly of the related variable y [$=+\sqrt{f_C}$] called the "Nordheim parameter". All of these can be seen as derived from different mathematical approximations for v(x). The purpose of this Poster is to classify these different approximations for v(x), and indicate typical mathematical accuracies in the range $0.15\leq x \leq 0.45$, which is the "pass" range for f_C in the orthodoxy test [see R.G. Forbes, Proc. R. Soc. Lond. A **469**, 20130171 (2013)].

The main divisions of the classification are: (1) whether it is an "old" approximation, effectively based on a single power series, or a "new" approximation, effectively based on the (mathematically correct) use of two power series; (2) how many terms are used in the approximation.

It will be pointed out that the use of "old" approximations (which is still widespread in current literature) should now be regarded as obsolete, first because the "new" approximations are of higher accuracy, second because the "old" approximations do not lead to "best physics" in the analysis of experimental current-voltage data or in the on-going development of improved data-analysis theory.

COMPARISONS

1. Introduction

1.1 General background

In the original papers, expressions for what I now call the *principal field emission special mathematical function* "v" were given in terms of the scaled field f or (in older papers) the Nordheim parameter y [$=+\sqrt{f}$]. It is now known that the function "v" is in fact a very special solution of the Gauss Hypergeometric Differential

Equation (HDE). Hence, best mathematical procedure is to express "v" as a function of the independent variable in the Gauss HDE. I call this the *Gauss variable* and denote it by the symbol x. Thus, I write "v(x)". In accordance with the usual international convention for typesetting special mathematical functions, I now typeset the symbol "vee" upright.

More generally, I am now making a distinction between the pure mathematics of the function v(x) and the applications of this function in modelling (which exist in several different contexts). In order to apply this function to the kind of modelling that exists in the context of the theory of field electron emission (FE) *from metals*, one either sets $x=f$ or sets $x=y^2$. The author's view is that setting $x=f$ is normally the most useful procedure, especially when discussing current-voltage characteristics, but that setting $x=y^2$ can be useful in theoretical derivations that involve integrations with respect to the energies of electron states.

Since this is a document about mathematical approximations, all formulae for "vee" that are expressed elsewhere in terms of y or in terms of f have been converted here to be formulae in terms of x .

1.2 Range of interest

In the author's "orthodoxy test" [1], it is assumed (based on multiple experimental analyses) that room-temperature field electron emitters normally operate within the scaled-field range $0.15<f<0.45$. Comparisons of accuracy are therefore made here, as in [2], for this range. The comparisons are made with the results from the "high-precision" approximation, which has itself been validated by comparing results with "exact" values obtained by means of MAPLE™ computer-algebra evaluations of a definition of v(x) in terms of complete elliptic integrals. These MAPLE™ evaluations are capable of yielding results correct to at least about 30 decimal places.

2. Approximations

2.1 The most commonly used "old" (now obsolete) approximations

The most commonly used "old" approximations (are shown in Table 1. A wider comparison, involving older approximations now less frequently used, was made in [2]. All these "old" approximations have now been made obsolete by the (more accurate) modern Forbes-Deane approximation.

Table 1: Comparison of approximations for v(x).

Name, date and & reference	Mathematical form	Maximum absolute error in $0.15\leq x\leq 0.45$	Maximum relative error in $0.15\leq x\leq 0.45$
Old approximations			
Charbonnier-Martin (1962) [3]	$\mathrm{v}(x)\approx 0.956-1.062x$	0.011	2.2%
Elinson-Shrednik (1974) [4]	$\mathrm{v}(x)\approx 0.95-1.03x$	0.0047	0.59%
Spindt et. (1976) [5]	$\mathrm{v}(x)\approx 0.95-x$	0.013	2.4%
21st-Century approximations			
Forbes-Deane (2006) [6] ("Simple good approximation")	$\mathrm{v}(x)\approx 1-x+(1/6)x\ln x$	0.0024	0.33%
High-precision approximation (2007) [7]	See below.	$< 8\times10^{-10}$	not relevant

2.2 The 21st-Century expressions for v(x)

As discussed in the main presentation for this conference [8], the mathematical basis for the so-called "21st-Century" approximations is knowledge of the form of the exact series expansion for the special mathematical function v(x) [9, but replace the symbol l' used there by the symbol x now preferred]. It can be shown that this series expansion can be put in the slightly modified but equivalent form shown in Table 2.

Table 2: To show the forms of the "21st-Century" expressions for v(x).	
Name	*Form*
Exact series expansion (modified form)	$\mathrm{v}(x) \equiv (1-x)\{1+P_\infty(x)\} + (x\ln x)\cdot Q_\infty(x)$
High-precision (HP) approximation	$\mathrm{v}(x) \cong (1-x)\{1+P_{\mathrm{HP}}(x)\} + (x\ln x)\cdot Q_{\mathrm{HP}}(x)$
"Simple good approximation"	$\mathrm{v}(x) \cong (1-x) + (x\ln x)\cdot(1/6)$

In Table 2, $P_\infty(x)$ and $Q_\infty(x)$ are two infinite power series that can be derived from the two infinite power series defined in [7]. $P_{\mathrm{HP}}(x)$ and $Q_{\mathrm{HP}}(x)$ are two four-term power series defined in the following way.

$$P_{\mathrm{HP}}(x) \equiv \sum_{i=1}^{4} p_i x^i, \qquad Q_{\mathrm{HP}}(x) \equiv \sum_{i=1}^{4} q_i x^{i-1},$$

where the coefficients p_{i} and q_{i} are given in the following table.

i	p_i	q_i
1	0.032 705 304 46	0.187 499 344 1
2	0.009 157 798 739	0.017 506 369 47
3	0.002 644 272 807	0.005 527 069 444
4	0.000 089 871 738 11	0.001 023 904 180

These values were derived by numerical fitting to the computer-algebra evaluations of v(x) as described above.

2.3 Physical implications

In the exact mathematics, and related approximations, the existence of the term in "$x\ln x$" has an important physical implication. When a related FE equation is expressed in Fowler-Nordheim or Murphy-Good coordinates, then the term in $x\ln x$ modifies the form of the *pre-exponential*, and hence affects the value of the "formal emission area" extracted from the related plot. Thus, it is likely that—in order to develop experiment-based FE science—much greater attention needs to be paid to the precise measurement and physical interpretation of the parameter "formal emission area" ($A_{\mathrm{f}}^{\mathrm{SN}}$). In particular, the Forbes-Deane approximation allows the introduction of "Murphy-Good plots" [10], which allow more precise measurement of $A_{\mathrm{f}}^{\mathrm{SN}}$ than Fowler-Nordheim plots.

References

[1] R.G. Forbes, Proc. R. Soc. Lond. A **469**, 20130271 (2013).
[2] R.G. Forbes & J.H.B. Deane, J. Vac. Sci. Technol. B **28**, C2A33 (2010).
[3] F.M. Charbonnier & E.E. Martin, J. Appl. Phys. **33**, 1897 (1962).
[4] V.N. Shrednik, "Theory of field electron emission", Chap. 6 in: M.I. Elinson (ed.), *Unheated Cathodes* ('Soviet Radio', Moscow, 1974) (in Russian), see eq. (6.10).
[5] C.A. Spindt, I. Brodie, L. Humphrey & J.R. Westerberg, J. Appl. Phys. **47**, 5248 (1976).
[6] R.G. Forbes, Appl. Phys. Lett. **89**, 113122 (2006).
[7] R.G. Forbes & J.H.B. Deane, Proc. R. Soc. Lond. A **463**, 2907 (2007).
[8] R.G. Forbes, "Progress in reshaping field electron emission theory for the benefit of experimental scientists and engineers", pdf-file presentation, this conference.
[9] J.H.B. Deane & R.G. Forbes, J. Phys. A: Math. Theor. **41**, 395301 (2008).
[10] R.G. Forbes, R. Soc. Open Sci. **6**, 190912 (2019).